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Statistics and quantum chaos

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Abstract. We use multi-time correlation functions of quantum systems to construct random variables with statistical properties that reflect the degree of complexity of the underlying quantum dynamics.

1. Introduction

In the field of quantum chaos, the structure of time-correlation functions is of the utmost importance to study relaxation phenomena, to single out the existence of different timescales and to perform the semiclassical analysis [1]. In this paper, we look at time averages of multi-time correlation functions as expectations of particular random variables and suggest that their statistical properties might reflect the degree of irregularity of the quantum dynamics. In particular, on the level of the fluctuations of these random variables, a variety of different statistics seems likely to emerge, among them the semicircle distribution typical of random matrix theory, corresponding to different degrees of randomness.

2. Statistics of correlation functions

We shall in general be interested in the large-time behaviour of time correlation functions of quantum dynamical systems. For sake of comparison and broader generality, such a matter is better described in the more general setting of quantum and classical dynamical systems.

In quantum mechanics, one usually works with operators X on a Hilbert space $\mathcal{H}$ and the Heisenberg evolution generated by some Hamiltonian H

$$X \mapsto X(t) := \mathrm{e}^{\mathrm{i}Ht} X \mathrm{e}^{-\mathrm{i}Ht}. \tag{1}$$

Then, time-invariant expectations $X \mapsto \langle X \rangle := \langle \psi | X | \psi \rangle$ are computed by means of a suitable 'reference state' $H|\psi\rangle = 0$. In general, i.e. in the case of discrete quantum dynamics, $\exp(\mathrm{i}Ht)$ is replaced by the power U^n of a unitary operator U, with $U|\psi\rangle = |\psi\rangle$. Then, (1) brings about the time evolution up to an integer multiple $t = Tn$ of a unit of time T.

In classical mechanics, one has a phase space $\mathcal{X}$, a dynamical (Hamiltonian) flow connecting phase points (q, p) through trajectories (q_t, p_t) and a time-invariant, normalized phase-density distribution ρ, e.g. the canonical ensemble. It is, however, convenient to

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adopt an algebraic description based on the Koopman construction [2], using complex-valued functions f on $\mathcal{X}$ evolving in time according to

$$f(q, p) \mapsto f_t(q, p) := f(q_t, p_t). \tag{2}$$

Time-invariant expectations are obtained by averaging with respect to ρ

$$\langle f \rangle := \int_{\mathcal{X}} \mathrm{d}q\, \mathrm{d}p\, \rho(q, p) f(q, p). \tag{3}$$

The analogy between classical and quantum systems can be pushed further by considering the Hilbert space of square integrable functions f on phase space $\mathcal{X}$

$$\langle f | f \rangle := \int_{\mathcal{X}} \mathrm{d}q\, \mathrm{d}p\, \rho(q, p) |f(q, p)|^2 < \infty. \tag{4}$$

The classical observables act on this Hilbert space as multiplication operators: $f|g(q, p)\rangle := |f(q, p)g(q, p)\rangle$ and we can compute the expectation in (3) as the average of such an f with respect to the constant 'wavefunction' 1 on $\mathcal{X}$: $\langle f \rangle = \langle 1|f|1\rangle$.

In this way, classical and quantum systems can be treated on the same footing. Formally, the only difference is the algebra considered: commutative in the first case, non-commutative in the second. Nevertheless, this has quite profound consequences on the probabilistic structure of the theory. In fact, one of the difficult problems for truly quantum systems is to understand the implications of positivity, i.e. $\langle X^\dagger X \rangle \geqslant 0$, on the structure of the expectations. Indeed, the algebraic formulation of both classical and quantum dynamical systems indicates a possible way to extrapolate from the classical to the quantum context, but, at the same time, puts into evidence the differences between the two. For instance, the notion of mixing is expressed for both classical and quantum systems by the decorrelation property

$$\lim_t \langle XY(t)Z \rangle = \langle XZ \rangle \langle Y \rangle. \tag{5}$$

In classical dynamical systems, Z can be commuted over $Y(t)$ so that two observables X and Y suffice. Also, any classical correlation function as $\langle XY(t)ZU(t)VS(t)\rangle$, or the like, where the time t appears more than once, can be reduced to the above form. Because of a lack of commutativity this is not possible in quantum system, unless some form of asymptotic commutativity in time holds as, for instance,

$$\lim_t \langle [X, Y(t)]^* [X, Y(t)] \rangle = 0. \tag{6}$$

For infinite quantum systems, many properties can be deduced from (6) [3].

In most finite quantum systems, however, the (quasi-)energy spectrum is discrete and neither mixing, nor does asymptotic commutativity hold. Typically, in classically chaotic quantum systems, it is at this point that the notion of breaking-times appears [1]. We do not want to address this interesting topic here, but we will stick to quantum systems which are dynamically endowed with some degree of asymptotic clustering and show that different statistics of quantum random variables naturally emerge.

In the hierarchy of quantum clustering behaviours, stronger than mixing is multi-clustering [4]

$$\lim_{\min |t_a - t_b| \to \infty} \langle X^{(1)}(t_{i_1}) X^{(2)}(t_{i_2}) \ldots X^{(n)}(t_{i_n}) \rangle = \prod_{\ell=1}^{s} \left\langle \overrightarrow{\prod_{j \in J_\ell}} X^{(j)} \right\rangle \tag{7}$$

where J_ℓ is the subset $\{j_1, j_2, \ldots\}$ of $\{1, 2, \ldots, n\}$ such that $t_{j_i} = t_{j_k}$, that is we allow the same time to appear more than once, so that the number of different times s may be smaller than the number n of subindices. The shorthand notation $\min |t_a - t_b| \to \infty$ means that we let all differences between different times go to infinity and, finally, the arrow over the product means that the factors inside, which do not in general commute, have to appear in the same order as in the correlation function. It is not difficult to show that (7) is equivalent to having both asymptotic commutativity (6) and mixing (5) [5].

However, rather than in a situation where (6) holds, we are interested in multi-time correlation functions of the form

$$\langle X^{(1)}(t_{i_1}) X^{(2)}(t_{i_2}) \ldots X^{(n)}(t_{i_n}) \rangle \tag{8}$$

where $t_{i_1} \neq t_{i_2} \neq \cdots \neq t_{i_n}$, but, possibly, $t_{i_j} = t_{i_k}$ when $|j-k| > 1$. Indeed, one might rightly suspect that, precisely because of the possible irregularity of the dynamics, no asymptotic commutativity is available to simplify multi-time correlation functions. In this case, given several products (monomials) of observables at different times, some of them possibly equal, one cannot help but keep the monomials as they are. The only sensible algebraic operation left, apart from linear combinations and taking adjoints, is the concatenation of monomials into larger ones.

Concretely, let $X^{(1)}, X^{(2)}, \ldots, X^{(n)}$ be n observables from a suitable operator algebra $\mathcal{A}$ at time $t = 0$. Because of the time evolution, in spite of possible algebraic relations between $X^{(k)}$ and $X^{(\ell)}$ at $t = 0$, no simplifying commutation relations, e.g. commutativity, need to hold between $X^{(k)}(t_{i_k})$ and $X^{(\ell)}(t_{i_\ell})$ for large $|t_{i_k} - t_{i_\ell}|$. In general, one will have to cope with expectations of monomials of the form $X^{(1)}(t_{i_1}) X^{(2)}(t_{i_2}) \ldots X^{(n)}(t_{i_n})$, the observables at time $t = 0$ having evolved up to a set of times $t_{i_1}, t_{i_2}, \ldots, t_{i_n}$, some of them possibly equal, without much room for simplifications. Thus, the only sensible algebraic setting is that provided by a 'free product' [6] of copies of the algebra $\mathcal{A}$ consisting of (linear combinations of) monomials $X^{(1)}_{i_1} X^{(2)}_{i_2} \ldots X^{(n)}_{i_n}$, the subindex i_ℓ locating the observable $X^{(\ell)}$ within the i_ℓth copy of $\mathcal{A}$, with the following rules.

(a) Whenever the identity appears it can be dropped.

(b) Whenever two consecutive observables $X^{(k)}_{i_k}$ and $X^{(k+1)}_{i_{k+1}}$ carry equal subindices ($i_k = i_{k+1}$), then they must be considered as the single observable $(X^{(k)} X^{(k+1)})_{i_k}$.

We stress that in the asymptotic free algebra, monomials are multiplied by concatenation without any simplification rule between consecutive letters except for the previous requests (a) and (b).

We shall call the algebra constructed above the 'asymptotic free algebra' and denote it by $\mathcal{A}_\infty$. Furthermore, we define an expectation functional $\langle \cdot \rangle_\infty$ on the monomials $X^{(1)}_{i_1} X^{(2)}_{i_2} \ldots X^{(n)}_{i_n}$ by computing consecutive multi-time averages of correlation functions as in (8), namely

$$\langle X^{(1)}_{i_1} X^{(2)}_{i_2} \ldots X^{(n)}_{i_n} \rangle_\infty := \lim_{T_s} \ldots \lim_{T_1} \frac{1}{T_1 \ldots T_s} \sum_{t_s=0}^{T_s} \ldots \sum_{t_1=0}^{T_0} \langle X^{(1)}(t_{i_1}) \ldots X^{(n)}(t_{i_n}) \rangle \tag{9}$$

in the case of discrete time dynamical systems, otherwise sums have to be replaced by integrals. In the expression above, all time indices $t_{j_\ell} \in \{t_{i_1}, t_{i_2}, \ldots, t_{i_n}\}$ such that $t_{i_\ell} = t_j$ contribute to the single time average with respect to t_j. The index s just counts the number of different times that appear in the multi-time correlation function to be averaged as in (9). These expectations return positive values when used to compute expectations of positive operators and thus the left-hand side member of (9) allows for a consistent probabilistic interpretation [5]. Notice that, according to the definition, given an observable $X \in \mathcal{A}$, $\langle X_{i_\ell} \rangle_\infty = \langle X \rangle$, whatever the location in a i_ℓ copy of $\mathcal{A}$ contributing to the asymptotic free algebra $\mathcal{A}_\infty$.

As a first application, let us assume that multi-clustering (7) holds. Then, (9) can be readily computed yielding

$$\langle X^{(1)}_{i_1} X^{(2)}_{i_2} \cdots X^{(n)}_{i_n} \rangle_\infty = \prod_{\ell=1}^{k} \Bigg\langle \overrightarrow{\prod_{j\in J_\ell}} X^{(j)} \Bigg\rangle_\infty = \prod_{\ell=1}^{s} \Bigg\langle \overrightarrow{\prod_{j\in J_\ell}} X^{(j)} \Bigg\rangle.$$

We present explicitly a few expectations choosing for notational simplicity $X^{(1)} = A$, $X^{(2)} = B$ and so on. Remember that the subscripts refer to the times with respect to which the limits in (9) are computed, so that equal subscripts mean that equal times have been considered:

$$\begin{aligned} &\langle A_1 \rangle_\infty = \langle A \rangle \\ &\langle A_1 B_2 \rangle_\infty = \langle A \rangle \langle B \rangle \\ &\langle A_1 B_2 C_1 \rangle_\infty = \langle AC \rangle \langle B \rangle \\ &\langle A_1 B_2 C_1 D_2 \rangle_\infty = \langle AC \rangle \langle BD \rangle. \end{aligned} \tag{10}$$

Notice that, because of (6), we could consistently impose that observables pertaining to different copies of $\mathcal{A}$ in the asymptotic free algebra $\mathcal{A}_\infty$ commute, that is $[X_k, Y_\ell] = 0$ for $k \neq \ell$. Moreover, the type of clustering into expectations of smaller monomials in (10) is an expression of statistical independence of observables well separated in time. This has the important consequence that, according to the central limit theorem, when $N \to \infty$, the fluctuations $1/N \sum_{j=1}^{N} \tilde{A}_j$ in the asymptotic free algebra $\mathcal{A}_\infty$ of centred observables $A \in \mathcal{A}$ become Gaussian random variables [7].

A totally different notion of statistical independence for observables belonging to the asymptotic free algebra, called 'free independence' or 'freeness', is defined by the following decoupling scheme [6]

$$\langle X^{(1)}_{i_1} X^{(2)}_{i_2} \ldots X^{(n)}_{i_n} \rangle_\infty = 0 \qquad \text{if } \langle X^{(\ell)} \rangle_\infty = \langle X^{(\ell)} \rangle = 0 \tag{11}$$

with $\ell = 1, \ldots, n$ and $i_1 \neq i_2 \neq \ldots \neq i_n$. As a comparison with (10), we write the first few expectations under the assumption that they exhibit free independence. We let $X^{(1)} = A$, $X^{(2)} = B$ and so on and use (11) after writing, say A, as $A = \langle A \rangle \mathbf{1} + \tilde{A}$ where $\tilde{A}$ is now centred. Then, we notice that, if $A \in \mathcal{A}$ is centred for $\langle \cdot \rangle$, it is also centred for $\langle \cdot \rangle_\infty$ as an observable of the asymptotic free algebra $A \in \mathcal{A}_\infty$. Thus,

$$\begin{aligned} &\langle A_1 \rangle_\infty = \langle A \rangle \\ &\langle A_1 B_2 \rangle_\infty = \langle A \rangle \langle B \rangle \\ &\langle A_1 B_2 C_1 \rangle_\infty = \langle AC \rangle \langle B \rangle \\ &\langle A_1 B_2 C_1 D_2 \rangle_\infty = \langle AC \rangle \langle B \rangle \langle D \rangle + \langle A \rangle \langle BD \rangle \langle C \rangle - \langle A \rangle \langle B \rangle \langle C \rangle \langle D \rangle. \end{aligned} \tag{12}$$

It follows that the notion of free independence is incompatible with the usual statistical independence: $\langle A_1 B_2 A_1^* B_2^* \rangle_\infty = \langle AA^* \rangle \langle BB^* \rangle$ in the usual case, while $\langle A_1 B_2 A_1^* B_2^* \rangle_\infty = 0$ in the 'free' case, A and B being centred observables. As a consequence, the fluctuations of centred observables are no longer Gaussian random variables, but semicircularly distributed [6].

3. Examples

We consider a class of dynamical systems described by operators $e(t)$ at discrete times $t \in \mathbb{Z}$, $e(t)$ being a unitary operator $e = e^\dagger$, $e^2 = \mathbf{1}$, specified at time $t = 0$ and evolved up to time t according to an underlying quantum evolution. Since we are only interested in the essential features of the time evolution, like regularity or randomness, we do not take into account its detailed structure, but rather resort to a schematic description. We shall assume that the dynamics may be described by a so-called 'bit-stream' [8–10], that is by a sequence $a(1), a(2), \ldots$ of zeros and ones fixing the commutation relations between operators at different times $s, t = 1, 2, \ldots$

$$e(s+t)e(s) = (-1)^{a(t)} e(s)e(s+t). \tag{13}$$

Obviously, these commutation relations strongly depend on the statistical properties of the bit-stream.

The algebra $\mathcal{A}$ of observables of the system consists of linear combinations of monomials $w(\boldsymbol{t})$ of operators $e(t)$ of the form

$$w(\boldsymbol{t}) := e(t_{i_1})e(t_{i_2})\ldots e(t_{i_n}) \qquad \boldsymbol{t} = (t_{i_1}, t_{i_2}, \ldots, t_{i_n}). \tag{14}$$

By using the commutation relation (13) and the fact that $e(t)^2 = \mathbf{1}$, we may always assume that t is an ordered multi-index, i.e. $t_{i_1} < t_{i_2} < \cdots < t_{i_n}$. The probabilities of selfadjoint monomials $w(\boldsymbol{t})$ are specified by the expectations $\langle w(\boldsymbol{t})\rangle$ with respect to a given state $\langle\cdot\rangle$.

If there are no preferred observables to single out apart from the identity, a meaningful statistic arises from

$$\langle w(\boldsymbol{t})\rangle = 0 \qquad \langle \mathbf{1}\rangle = 1. \tag{15}$$

The dynamics during a single timestep is given by the shift on the indices of the operators $e(t)$:

$$w(\boldsymbol{t}) \mapsto w(\boldsymbol{t}+1) := e(t_{i_1}+1)e(t_{i_2}+1)\ldots e(t_{i_n}+1). \tag{16}$$

In spite of the extreme simplicity, the variety of statistics brought about by the expectations in (15) together with the bit-streams is nevertheless noticeable [5]. Notice that $\langle w(\boldsymbol{t})\rangle$ can appropriately be called a multi-time correlation function for the dynamics given in (16).

3.1. Free shift

We shall now consider the so-called 'free shift'. In its most basic form it is a quantum shift, but without any algebraic relations as in (13), so that the only possible simplification in products of observables comes from $e^2 = \mathbf{1}$. It is rather obvious that system observables do not commute, even when largely separated in time. The statistics of correlation functions is now described by 'free independence', that is by (11). In order to prove the assertion, we observe that, because of (15), general centred observables $\tilde{A}$, i.e. $\langle \tilde{A}\rangle = 0$, are obtained by linear combinations of monomials. Since we want to compute expectations of the form $\langle w^{(1)}_{i_1} w^{(2)}_{i_2} \ldots w^{(n)}_{i_n}\rangle_\infty$, where $i_j \neq i_{j+1}$ for all $j = \{1, 2, \ldots, n\}$, we consider time limits

$$\lim_{\min |t_a - t_b| \to \infty} \langle w^{(1)}(t_{i_1}) w^{(2)}(t_{i_2}) \ldots w^{(n)}(t_{i_n})\rangle \tag{17}$$

where the $w^{(j)}$ are centred monomials as in (14) and $w^{(j)}(t_{i_j})$ are the evolved ones up to times t_{i_j} according to (16) and $t_{i_j} \neq t_{i_{j+1}}$. It is then clear that, for sufficiently large differences between any two consecutive times, there cannot be simplifications due to the rule $e(t)^2 = \mathbf{1}$. Therefore, because of (15), expectations of products of observables as in (17) will vanish in the limit, whence $\langle w^{(1)}_{i_1} w^{(2)}_{i_2} \ldots w^{(n)}_{i_n}\rangle_\infty = 0$.

3.2. Regular and irregular quantum shifts

Quantum shifts governed by generic bit-streams present intermediate situations interpolating between the case of asymptotic commutativity and the total absence of algebraic relations between observables largely separated in time. In fact, one easily calculates

$$\langle [e(t), e(s)]^*[e(t), e(s)] \rangle = (1 - (-1)^{a(|t-s|)})^2$$

so that, unless the bit-stream is regular and $\lim_t a(t) = 0$ or 1, there are no definite commutation relations among operators largely separated in time. If $a(t)$ is eventually vanishing, then the quantum shift is asymptotically Abelian and we expect the usual statistical independence for the (random variables) observables of $\mathcal{A}_\infty$. If $a(t)$ tends to 1, we obtain Fermionic independence. Otherwise, the observables of $\mathcal{A}$ do not asymptotically commute and in connection with the degree of irregularity of the bit-stream, one expects typical statistics to be exhibited in $\mathcal{A}_\infty$. Yet, no irregularity is enough to enforce free independence. In fact, let us consider the simple observable $w = e \in \mathcal{A}$ and monomials constructed with alternating products of $e(t_1)$ and $e(t_2)$, $t_1 \neq t_2$. The first, possibly non-zero, expectation of such monomials is

$$\langle e(t_1)e(t_2)e(t_1)e(t_2) \rangle = (-1)^{a(|t_2-t_1|)}.$$

Since $\langle e_i \rangle_\infty = \langle e \rangle = 0$ for all i, freeness demands $\langle e_1e_2e_1e_2 \rangle_\infty = 0$. By choosing a sufficiently irregular bit-stream, e.g. a typical path of an unbiased Bernoulli process, we can enforce

$$\langle e_1e_2e_1e_2 \rangle_\infty = \lim_{T_1,T_2\to\infty} \frac{1}{T_1T_2} \sum_{t_1=0}^{T_1} \sum_{t_2=0}^{T_2} (-1)^{a(|t_2-t_1|)} = 0. \tag{18}$$

Nevertheless, using (13) one easily deduces

$$\langle e(t_1)e(t_2)e(t_1)e(t_2)e(t_1)e(t_2)e(t_1)e(t_2) \rangle = 1$$

whence $\langle e_1e_2e_1e_2e_1e_2e_1e_2 \rangle_\infty = 1$, whereas freeness would amount to the vanishing of that expectation, too.

Notice that $\langle (e_1e_2)^4 \rangle_\infty$ is the first expectation of alternating products not to vanish. By reducing the degree of irregularity of the bit-stream, one may make $\langle (e_1e_2)^2 \rangle_\infty \neq 0$ in (18). This should be compared with (10) and (12) fixing $C = A$, $D = B$ with A and B centred observables. Of course, the full statistics needs the study of higher moments. However, one may already guess the connection between the irregularity of the quantum dynamics and the clustering of expectations of higher monomials: the more the randomness the less the contributions.

3.3. Quantum Koopmanism

As a somewhat different model, we consider a classical flow $(q, p) \mapsto (q_t, p_t)$ with mixing properties on phase space, namely

$$\lim_{t\to\infty} \langle f | g_t \rangle = \lim_{t\to\infty} \int_{\mathcal{X}} \mathrm{d}q\, \mathrm{d}p\, \rho(q, p) \overline{f(q, p)} g(q_t, p_t) = \langle f \rangle \langle g \rangle$$

where the Koopman Hilbert space description (4) for classical systems has been used. We now proceed to a 'non-canonical' quantization whereby the quantum evolution of 'wavefunctions' is exactly the classical one [11]. Given functions f, g in the Koopman Hilbert space, operators of the form $|f\rangle\langle g|$ may be used to construct the non-commutative algebra of all finite rank operators. Expectations of observables $|f\rangle\langle f|$ are given by

$\langle |f\rangle\langle f|\rangle = |\langle 1|f|1\rangle|^2$. Finally, the dynamics shifts $|f\rangle\langle f|$ into $|f_t\rangle\langle f_t|$ with f_t as in (2). One can then deduce that

$$\lim_{t\to\infty}\langle R(f)R(g_t)\rangle = \langle R(f)\rangle\langle R(g)\rangle$$

where $R(f) := |f\rangle\langle f|$. The above product structure extends to the set of finite rank matrices $A, B, \ldots, F$ and multi-clustering as in (7) holds, namely

$$\lim_{\min|t_a-t_b|\to\infty}\langle 1|A(t_{i_1})B(t_{i_2})\ldots F(t_{i_n})|1\rangle = \langle A\rangle\langle B\rangle\ldots\langle F\rangle.$$

In contrast to (7), there is no clustering of operators carrying the same time-index.

The above limits can be used to construct the asymptotic state $\langle\cdot\rangle_\infty$ on the asymptotic free algebra $\mathcal{A}_\infty$. Explicitly

$$\langle A_1B_2\ldots F_n\rangle_\infty = \langle A\rangle\langle B\rangle\ldots\langle F\rangle.$$

Notice that the identity operator **1** is not a finite rank matrix and, in order to construct centred observables $\tilde{A} := A - \langle A\rangle\mathbf{1}$, one has to add it to the finite rank operators. Such a dynamical system is neither commutative, nor asymptotically commutative and therefore, the usual statistical independence (7) and thus a Gaussian distribution of fluctuations is not expected to hold. Freeness does not show up either as a property of the asymptotic free algebra. Indeed, considering centred observables $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$, one can prove that

$$\begin{aligned}
&\langle\tilde{A}_1\rangle_\infty = 0\\
&\langle\tilde{A}_1\tilde{B}_2\rangle_\infty = 0\\
&\langle\tilde{A}_1\tilde{B}_2\tilde{C}_1\rangle_\infty = \langle A\rangle\langle B\rangle\langle C\rangle - \langle B\rangle\langle AC\rangle\\
&\langle\tilde{A}_1\tilde{B}_2\tilde{C}_1\tilde{D}_2\rangle_\infty = 0.
\end{aligned}\tag{19}$$

Unlike in (10) and (12) when we use centred observables, in (19) the first non-vanishing moment is already the third one, which somehow indicates that, despite the mixing property of the underlying classical dynamics which is carried over to an exotic quantum dynamics, the statistics on the asymptotic free algebra does not come nearer to the irregular quantum shifts discussed above. Interestingly, the previous way of extending a property of the classical time evolution, in this case phase space mixing, to a 'quantum' system was proposed in [11] to provide a counterexample to the claimed incompatibility between chaos and quantum mechanics. Subsequently, a physical application of these ideas was given in [12].

4. Conclusions

Usual statistical independence is a workable property in the context of infinitely extended dynamical systems appearing in statistical mechanics where a more or less strong degree of asymptotic commutativity is expected. However, when no asymptotic commutativity is available, the knowledge that multi-time correlation functions, with strictly ordered times such as in (7), cluster, is not sufficient to draw any conclusion about correlation functions where equal times appear as in (8). From the above examples, we learn that increasing random behaviours bring us closer to freeness in the sense that more and more asymptotic expectations vanish. This is particularly evident for quantum shifts, where regular bit-streams would make a lot of multi-time averages return non-zero values. On the other hand, free independence requires that all expectations of monomials of centred observables in the asymptotic free algebra vanish. This amounts to a total lack of any algebraic structure between observables at different times which is difficult to implement by means of any irregular bit-stream. However, freeness seems more likely on the level of the fluctuations in the asymptotic free algebra of sufficiently random quantum systems [5].

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References

[1] Casati G and Chirikov B 1995 *Quantum Chaos* (Cambridge: Cambridge University Press)
[2] Reed M and Simon B 1972 *Methods of Modern Mathematical Physics, I: Functional Analysis* (New York: Academic)
[3] Thirring W 1983 *A Course in Mathematical Physics, IV: Quantum Mechanics of Large Systems* (New York: Springer)
[4] Thirring W and Narnhofer H 1994 *Lett. Math. Phys.* **30** 307
[5] Andries J, Benatti F, De Cock M and Fannes M in preparation
[6] Voiculescu D V, Dykema K J and Nica A 1991 *Free Random Variables* (Providence, RI: American Mathematical Society)
[7] Goderis D, Verbeure A and Vets P 1989 *Prob. Theor. Relat. Fields* **82** 527
[8] Price G L 1987 *Can. J. Math.* **39** 492
[9] Powers R T 1988 *Can. J. Math.* **40** 86
[10] Størmer E 1992 *Inv. Math.* **110** 63
[11] Berry M 1990 True quantum chaos? An instructive example *Proc. Yukawa Symp. (Tokyo)*
[12] Pasmanter R A 1990 *Phys. Rev.* A **42** 3622